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Explicit solution of the incompressible Navier–Stokes equations with linear finite elements

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The authors wish to dedicate this paper to Roland Glowinski, on the occasion of his nomination for full membership of the Académie des Sciences de Paris.

Abstract

A three-field finite element scheme for the explicit iterative solution of the stationary incompressible Navier–Stokes equations is studied. In linearized form the scheme is associated with a generalized time-dependent Stokes system discretized in time. The resulting system of equations allows for a stable approximation of velocity, pressure and stress deviator tensor, by means of continuous piecewise linear finite elements, in both two- and three-dimensional space. Convergence in an appropriate sense applying to this finite element discretization is demonstrated, for the stationary Stokes system.

Keywords: Explicit solution; Linear finite elements; Navier–Stokes equations; Three-field formulation

1. Introduction

Many works have been dedicated to splitting algorithms for the time integration or the iterative solution of the incompressible Navier–Stokes equations. In the context of finite element discretizations, the contributions of Glowinski et al. (see e.g. [1] and references therein) are among the earliest successful ones. In this work a three-field algorithmic formulation, first designed to allow the explicit integration of the equations governing time-dependent viscoelastic flow (cf. [2]), is applied to the case of stationary flows of Newtonian incompressible fluids, in either two- or three-dimensional space. The three fields represented in the formulation are the velocity $\mathbf{u}$, the pressure p and the stress deviator tensor σ. The algorithm is based on the iterative solution of the system, by means of a projection algorithm inspired by the least squares pressure correction technique, introduced by Goldberg and Ruas in [3] for the time integration of the incompressible Navier–Stokes equations. The resulting solution scheme accommodates a space discretization by continuous piecewise linear interpolations of these three flow variables, provided a suitable variational formulation is employed. This leads to stable and accurate discrete counterparts. The fact that the unusual

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variable σ is explicitly represented in Newtonian flow equations is largely compensated by the algorithm in use: not only are the three fields uncoupled at every iteration, but also both $\mathbf{u}$ and σ may be determined node by node. p in turn is the solution of a suitable Poisson equation with fully consistent boundary conditions, which is one of the main advantages of the present approach, as compared to most projection algorithms in use.

2. Viscous flow equations

Let Ω be a bounded domain of $\mathbb{R}^N$, $N = 2$ or $N = 3$ and $\partial\Omega$ be its boundary with unit outer normal vector $\mathbf{n}$. Let also $\langle\cdot,\cdot\rangle_{1/2,\partial\Omega}$ denote the duality product between $H^{1/2}(\partial\Omega)^N$ and $H^{-1/2}(\partial\Omega)^N$ (cf. [4]), and $(\cdot,\cdot)$ denote the standard inner product of $[L^2(\Omega)]^M$, $M \in \mathbb{N}^*$ and $\|\cdot\|$ the associated norm.

We consider the stationary flow in Ω of an incompressible Newtonian viscous fluid, under the action of volumetric forces $\mathbf{f}$, assuming that its velocity $\mathbf{u}$ is fully prescribed on $\partial\Omega$, say $\mathbf{u} = \mathbf{g}$, where $\mathbf{g}$ fulfills $\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\mathrm{d}S = 0$. We assume that $\mathbf{f} \in L^2(\Omega)^N$, and $\mathbf{g} \in H^{1/2}(\partial\Omega)^N$. Then denoting the stress deviator tensor by σ, the hydrostatic pressure by p, and representing by ∇ the gradient of a scalar or a vector valued function, the flow is governed by the Navier–Stokes equations expressed in terms of $(\mathbf{u}, p, \sigma)$, namely,

$$\left.\begin{array}{l} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\sigma + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \\ \sigma = 2\eta D(\mathbf{u}) \end{array}\right\} \text{ in } \Omega. \tag{1}$$

In (1) η is the inverse of the Reynolds number, and $D(\cdot)$ is the symmetric gradient, i.e., $D(\cdot) := \left\{\nabla(\cdot) + [\nabla(\cdot)]^T\right\}/2$.

In the case of low Reynolds numbers we can legitimately linearize (1) into the classical (three-field) Stokes system, namely,

$$\left.\begin{array}{l} -\nabla\cdot\sigma + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \\ \sigma = 2\eta D(\mathbf{u}) \end{array}\right\}. \tag{2}$$

3. Fictitious time discretization

Our methodology is designed for the explicit iterative solution of system (1) or even of (2) after discretization in space as seen in Section 5. However in this section and in the next one we focus on the following semi-implicit algorithm for solving system (1). Let $\Delta t > 0$ be a fictitious time step and $\lambda \geq 0$ be a given parameter. Letting a field $\mathbf{u}^0$ satisfying $\mathbf{u}^0 = \mathbf{g}$ on $\partial\Omega$ and a deviatoric symmetric tensor σ^0 be initial guesses of $\mathbf{u}$ and σ, respectively, we successively determine approximations $(\mathbf{u}^m, p^m, \sigma^m)$ of $(\mathbf{u}, p, \sigma)$ for $m = 1, 2, \ldots$, prescribing $\mathbf{u}^m = \mathbf{g}$ on $\partial\Omega$, as the solution of the following system in Ω:

$$\left.\begin{array}{l} \dfrac{\mathbf{u}^m - \mathbf{u}^{m-1}}{\Delta t} + (\mathbf{u}^{m-1}\cdot\nabla)\mathbf{u}^{m-1} - \nabla\cdot\sigma^m + \nabla p^m = \mathbf{f} \\ \nabla\cdot\mathbf{u}^m = 0 \\ \sigma^m + \lambda\dfrac{\sigma^m - \sigma^{m-1}}{\Delta t} = 2\eta D(\mathbf{u}^m) \end{array}\right\}. \tag{3}$$

Throughout the sequel we assume that the Reynolds number is sufficiently low to allow for the linearization of (3) into the problem (4). The case of higher Reynolds numbers is more complex, but can be treated using arguments borrowed from other works, among which [5] is one of the most inspiring.

Then taking the same initial conditions $\mathbf{u}^0$ and σ^0 and boundary conditions for $\mathbf{u}^m$, we will deal with the system, for $m = 1, 2, \ldots$,

$$\left.\begin{array}{l} \dfrac{\mathbf{u}^m - \mathbf{u}^{m-1}}{\Delta t} - \nabla\cdot\sigma^m + \nabla p^m = \mathbf{f} \\ \nabla\cdot\mathbf{u}^m = 0 \\ \sigma^m + \lambda\dfrac{\sigma^m - \sigma^{m-1}}{\Delta t} = 2\eta D(\mathbf{u}^m) \end{array}\right\}. \tag{4}$$

Theorem 1. *For any choice of Δt, for any $\lambda > 0$ and for every m, problem (4) has a unique solution $(\mathbf{u}^m, p^m, \sigma^m) \in [H^1(\Omega)]^N \times L^2_0(\Omega) \times [L^2(\Omega)]^{N\times N}$, and its solution converges to the solution $(\mathbf{u}, p, \sigma)$ of (2) in the norm of $L^2(\Omega) \times L^2(\Omega)^N \times L^2(\Omega)^{N\times N}$ as m goes to infinity.*

Proof. The proof of existence and uniqueness is based on recurrence and on the fact that for every m problem (4) is nothing but a modified Stokes problem. For more details and for the proof of convergence as m goes to ∞ we refer the reader to [2]. □

We will next study (4) as an implicit algorithm for solving (2) iteratively. However for later convenience we consider instead the following alternative for determining p^m, $\mathbf{u}^m$ and σ^m, for $m = 1, 2, \ldots$, with $\mathbf{u}^m = \mathbf{g}$ on $\partial\Omega$, and assuming that $\mathbf{u}^{m-1}$ and σ^{m-1} are known:

$$\left.\begin{array}{ll} \Delta p^m - \nabla\cdot(\nabla\cdot\sigma^m) - \nabla\cdot\mathbf{u}^{m-1}/\Delta t = \nabla\cdot\mathbf{f} & \text{in } H^{-1}(\Omega) \\ \dfrac{\mathbf{u}^m - \mathbf{u}^{m-1}}{\Delta t} = \nabla\cdot\sigma^m - \nabla p^m + \mathbf{f} & \text{in } [L^2(\Omega)]^N \\ \sigma^m + \lambda\dfrac{\sigma^m - \sigma^{m-1}}{\Delta t} = 2\eta D(\mathbf{u}^m) & \text{in } [L^2(\Omega)]^{N\times N} \end{array}\right\}. \tag{5}$$

Remark 2. Assuming that in (5) σ^m is known, like for [3], a pressure Poisson equation formally holds with truly consistent non-homogeneous Neumann boundary conditions, derived from the multiplication of both sides of the first equation of (4) with $\mathbf{n}$ (cf. [2]).

Proposition 3.1. *Systems (5) and (4) are equivalent for every m.*

Proof. For this proof we refer the reader to [2]. □

4. Space discretization

In the remainder of this work we deal with a finite element discrete analogue of (4), assuming that Ω is either a polygon for $N = 2$ or a polyhedron for $N = 3$. Let then T_h be a partition of Ω into N-simplices with maximum edge length equal to h. We assume that T_h satisfies the usual compatibility conditions for finite element meshes, and that it belongs to a quasi-uniform family of partitions. For every $K \in T_h$ we further denote by $P_1(K)$ the space of polynomials of degree less than or equal to one defined in K. In so doing we introduce the following spaces or manifolds associated with T_h:

$$\begin{aligned} &S_h := \{v \mid v \in C^0(\bar{\Omega}) \text{ and } v|_K \in P_1(K),\ \forall K \in T_h\}, \\ &\mathbf{V}_h^0 := \mathbf{V}_h \cap H_0^1(\Omega)^N, \quad \text{with } \mathbf{V}_h := \{\mathbf{v} \mid \forall i\ v_i \in S_h\}, \\ &\mathbf{V}_h^{\mathbf{g}} := \{\mathbf{v} \in \mathbf{V}_h \mid \mathbf{v}(P) = \mathbf{g}(P)\ \forall \text{vertex } P \text{ of } T_h \text{ on } \partial\Omega\}, \\ &Q_h := S_h \cap L_0^2(\Omega), \\ &\Sigma_h := \{\tau \mid \tau \in [S_h]^{N\times N},\ \tau = \tau^T\}. \end{aligned}$$

We further define $\mathbf{u}_h^0$ to be the field of $\mathbf{V}_h^{\mathbf{g}}$ satisfying $\mathbf{u}_h^0(P) = \mathbf{u}^0(P)$, and σ_h^0 to be the tensor of Σ_h satisfying $\sigma_h^0(P) = \sigma^0(P)$, for every vertex P of T_h, and set for every $m, m = 0, 1, \ldots$:

$$\mathbf{u}_h^m := \mathbf{w}_h^m + \mathbf{u}_h^0,$$

where $\mathbf{u}_h^m$ is the approximation of $\mathbf{u}^m$ in $\mathbf{V}_h^{\mathbf{g}}$.

Finally, defining

$$\begin{aligned} a\left((p,\mathbf{u},\sigma),(q,\mathbf{v},\tau)\right) := {}& \Delta t^2\left(\nabla p - \nabla\cdot\sigma, \nabla q\right) + (\mathbf{u},\mathbf{v}) \\ & + \Delta t\left(\nabla p - \nabla\cdot\sigma, \mathbf{v}\right) + \frac{\lambda + \Delta t}{2\eta}(\sigma,\tau) + \Delta t^2\left(\nabla\cdot\sigma - \nabla p, \nabla\cdot\tau\right) \\ & \forall[(p,\mathbf{u},\sigma),(q,\mathbf{v},\tau)] \in [H^1(\Omega)\cap L_0^2(\Omega) \times L^2(\Omega)^N \times \mathbf{H}(\mathrm{div},\Omega)^N]^2 \end{aligned} \tag{6}$$

and

$$L_h\left((q,\mathbf{v},\tau)\right) := \Delta t^2\left(\mathbf{f}, \nabla q - \nabla\cdot\tau\right) + \Delta t\langle \mathbf{g}, (\tau - Iq)\mathbf{n}\rangle_{1/2,\partial\Omega} + \Delta t\left(\mathbf{f},\mathbf{v}\right)$$
$$+ (\mathbf{u}_h^{m-1},\mathbf{v}) + \Delta t(\mathbf{u}_h^{m-1}, \nabla q - \nabla\cdot\tau) + \frac{\lambda}{2\eta}(\sigma_h^{m-1},\tau) - (\mathbf{u}_h^0,\mathbf{v})$$
$$\forall (q,\mathbf{v},\tau)\in H^1(\Omega)\cap L_0^2(\Omega)\times L^2(\Omega)^N\times \mathbf{H}(\mathrm{div},\Omega)^N \tag{7}$$

we set the following problem to approximate (4) for every m, $m = 1, 2, \ldots$.

$$\left.\begin{array}{l}\text{Find } \left(p_h^m, \mathbf{w}_h^m, \sigma_h^m\right)\in Q_h\times \mathbf{V}_h^0\times \Sigma_h \quad \text{such that}\\ a\left(\left(p_h^m,\mathbf{w}_h^m,\sigma_h^m\right),(q,\mathbf{v},\tau)\right) = L_h\left((q,\mathbf{v},\tau)\right) \quad \forall (q,\mathbf{v},\tau)\in Q_h\times\mathbf{V}_h^0\times\Sigma_h\end{array}\right\}. \tag{8}$$

Now in order to derive convergence results for (8), we make the minimum regularity assumption:

*Assumption**: $p^m \in H^1(\Omega)$ for every m.

Notice that Assumption* should hold if Ω is convex (cf. [6]). Moreover it implies that σ^m belongs to $\mathbf{H}(\mathrm{div},\Omega)^N$ (cf. [7]).

Theorem 3. *For any λ and Δt and for every m, problem (8) has a unique solution. Moreover under* Assumption* *its solution converges to the one of (5) in the norm of $L^2(\Omega)\times[L^2(\Omega)]^N\times[L^2(\Omega)]^{N\times N}$ as h goes to* 0.

Proof. The proof of the existence and uniqueness issue is a straightforward application of the Lax–Milgram lemma and for the convergence result we apply Strang's inequality (cf. [8]). □

Finally combining Theorems 1 and 3, we can readily establish the following convergence result:

Theorem 4. *Let* Assumption* *hold. Then if $\lambda > 0$, as h goes to zero and m goes to ∞, the solution $(p_h^m, \mathbf{u}_h^m, \sigma_h^m)$ of (8) converges to the solution $(p, \mathbf{u}, \sigma)$ of (2) in $L^2(\Omega)\times L^2(\Omega)^N\times L^2(\Omega)^{N\times N}$.*

5. Explicit solution

To conclude we consider an uncoupling algorithm for solving explicitly system (8) at every iteration.

Set for every $m \ge 0$, $\sigma_h^{m,0} = \sigma_h^{m-1}$. Then for $s = 1, 2, \ldots$ determine approximations $p_h^{m,s}\in Q_h$, $\mathbf{w}_h^{m,s}\in\mathbf{V}_h^0$ and $\sigma_h^{m,s}\in\Sigma_h$ of p_h^m, $\mathbf{w}_h^m$ and σ_h^m by solving successively the following problems:

$$\left(\nabla p_h^{m,s},\nabla q\right) = (\mathbf{f}+\nabla\cdot\sigma_h^{m,s-1},\nabla q) + [(\mathbf{u}_h^{m-1},\nabla q) - \langle\mathbf{g}, q\mathbf{n}\rangle_{1/2,\partial\Omega}]/\Delta t \quad \forall q\in Q_h, \tag{9}$$

$$\left(\mathbf{w}_h^{m,s},\mathbf{v}\right) = \Delta t(\mathbf{f}+\nabla\cdot\sigma_h^{m,s-1} - \nabla p_h^{m,s},\mathbf{v}) + (\mathbf{u}_h^{m-1},\mathbf{v}) - (\mathbf{u}_h^0,\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{V}_h^0. \tag{10}$$

$$\frac{\lambda+\Delta t}{2\eta}\left(\sigma_h^{m,s},\tau\right) = \frac{\lambda}{2\eta}(\sigma_h^{m,s-1},\tau) - \Delta t^2(\mathbf{f}+\nabla\cdot\sigma_h^{m,s-1} - \nabla p_h^{m,s},\nabla\cdot\tau)$$
$$- \Delta t[(\mathbf{u}_h^{m-1},\nabla\cdot\tau) - \langle\mathbf{g},\tau\mathbf{n}\rangle_{1/2,\partial\Omega}] \quad \forall\tau\in\Sigma_h. \tag{11}$$

Remark 5. In practice problems (10) and (11) are solved node by node by the mass lumping technique, and convergence of the above algorithm is extremely fast, except for the very first values of m (cf. [9]).

Theorem 6. *For every m and for any $\lambda > 0$, the sequence of approximations $\{(p_h^{m,s}, \mathbf{u}_h^{m,s}, \sigma_h^{m,s})\}_s$ converges to $(p_h^m, \mathbf{u}_h^m, \sigma_h^m)$ as s goes to infinity in the sense of $L^2(\Omega)$, provided Δt fulfills $\Delta t \le h(2C)^{-1}\sqrt{\lambda/\eta}$, where C is a constant independent of h, for which the classical inverse inequality holds:*

$$\|\nabla\cdot\tau\| \le Ch^{-1}\|\tau\| \quad \forall\tau\in\Sigma_h.$$

Proof. For this proof we refer the reader to [9]. □

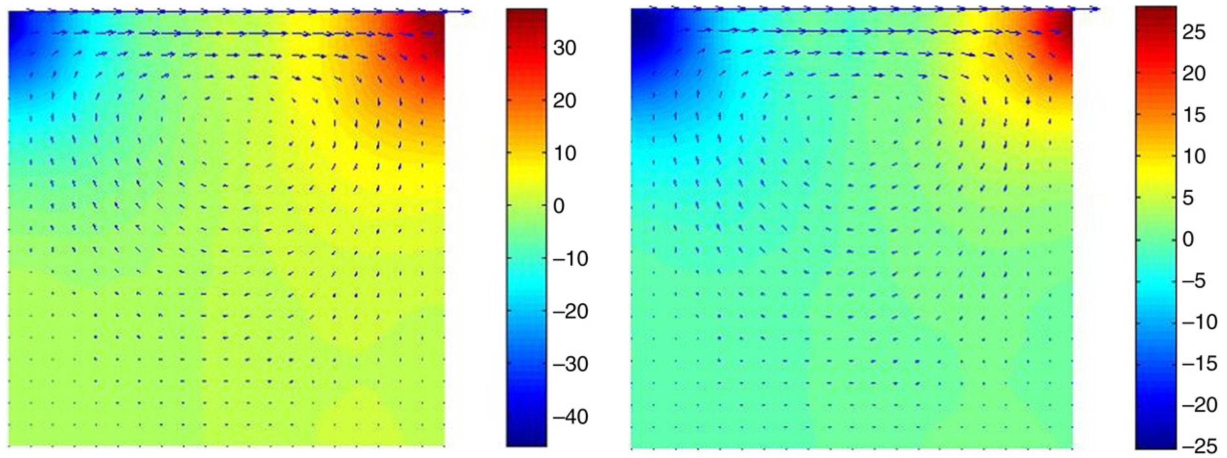

Fig. 1. Pressure and velocity field for $Re = 1$ and $Re = 100$.

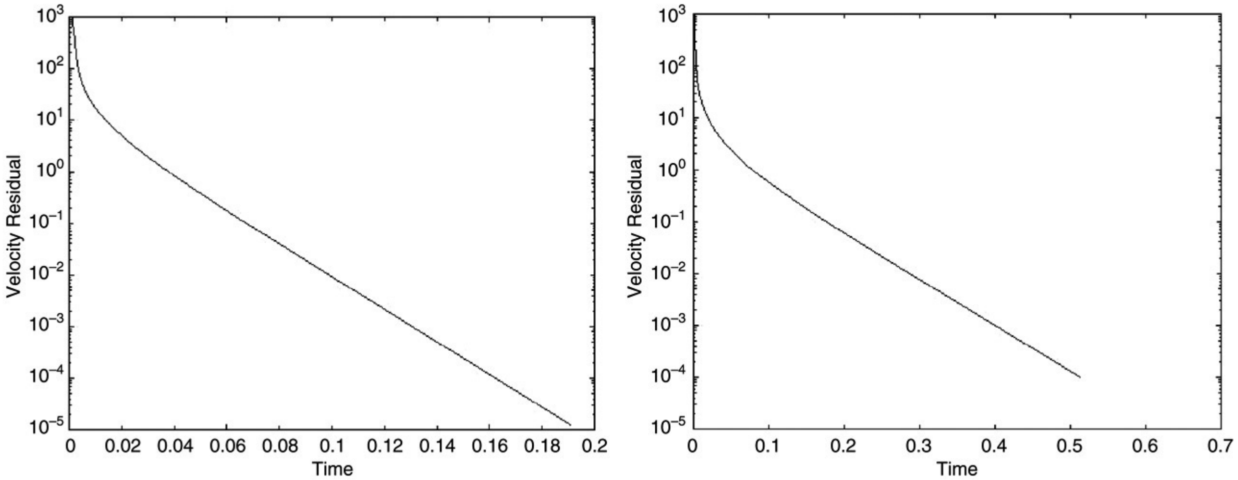

Fig. 2. Velocity error decay for $Re = 1$ and $Re = 100$.

6. Numerical results

The test case used to assess the above methodology is the classical lid-driven cavity flow. This case was first employed for code validation in [10] and since then has been used as a benchmark case in many references. This problem is defined in a square domain whose top edge is moving at a unit speed, while the other three edges of the square cavity are fixed walls with zero velocities. A zero-pressure constraint is applied in the middle of the bottom wall. A uniform mesh with 21×21 nodes is employed. $\Delta t = 0.001$ and $\lambda = 1.0$ were taken in the computations, for Reynolds numbers equal to 1 and 100. As one can observe from Fig. 1 our results are in quite good agreement with those of [10]. In Fig. 2 the maximum norm's decay of the modulus of the difference between two successive unknown velocity fields is displayed (the time indicated corresponds to the number of iterations multiplied by Δt).

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